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EULER STATISTICS OF THE DENSITY FIELD OF PASSIVE TRACERS IN INCOMPRESSIBLE AND COMPRESSIBLE MEDIA†

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General forms of a relation between the Lagrangian and Eulerian probability distributions of the Jacobian as well as the densities of passive tracers in a turbulent medium are obtained. Using these, in the cases of a compressible and an incompressible medium, the probability properties of the density under various initial distributions of the latter are analysed and studied. Expressions for the moments of the density field are obtained. © 1998 Elsevier Science Ltd. All rights reserved.

A number of papers have been devoted to the study of the statistical properties of the density of passive tracers in a turbulent medium (for example [1–4]). These papers typically investigate the mean density of passive tracers. However, its probability properties are not very well known. Note that even though the case of an incompressible medium is most frequently discussed in papers on turbulent diffusion, tracer diffusion in a compressible medium is also important in practice. This is due to the fact that buoyant tracers on the surface of an incompressible medium behave in a similar way to tracers in a two-dimensional compressible medium [5, 6]. The present paper is concerned with methods of describing the probability properties of passive tracers in incompressible and compressible media and with the study of their characteristic features.

1. GENERAL FORMULAE CONNECTING THE LAGRANGIAN AND EULERIAN STATISTICS OF FIELDS IN TURBULENT MEDIA

Below we devote considerable attention to comparing the statistics of random fields in the Lagrangian and Eulerian representations, as well as to establishing relations between similar statistical characteristics of turbulent media. These relations are useful in the theoretical analysis of chaotic hydrodynamic fields, when the analysis is carried out in one representation and a description of the properties of the fields in the other representation is required. The probability properties of the density $\rho(\mathbf{x}, t)$ of passive tracers in a random velocity field $\mathbf{v}(\mathbf{x}, t)$ are analysed. A full programme of such an investigation must involve a preliminary analysis of the statistical properties of the velocity field $\mathbf{v}(\mathbf{x}, t)$ satisfying non-linear hydrodynamic equations. However, this very complex problem has not been solved to date. Therefore, following for example [5, 6], we shall assume that the statistical properties of $\mathbf{v}(\mathbf{x}, t)$ are given. Namely, we shall assume that $\mathbf{v}(\mathbf{x}, t)$ is a statistically homogeneous isotropic Gaussian random field with correlation tensor

$$\langle v_i(\mathbf{x},t)v_j(\mathbf{x}+\mathbf{s},t+\tau)\rangle = b_{ij}(\mathbf{s},\tau) \tag{1.1}$$

There are fairly general relations between the Eulerian and Lagrangian statistical characteristics of random fields. Using these relations we can establish the Eulerian statistics of random fields from the known Lagrangian statistics. Therefore, we shall first derive and discuss these connecting formulae. To do this we consider the joint Eulerian probability density

$$f_E(\mathbf{y},j;\mathbf{x},t) = \langle\delta(\mathbf{y}-\mathbf{Y}(\mathbf{x},t))\delta(j-j(\mathbf{x},t))\rangle \tag{1.2}$$

of the Lagrangian coordinates $\mathbf{Y}(\mathbf{x}, t)$ of a particle which happens to be at the observation point with Eulerian coordinates $\mathbf{x}$, and the Eulerian Jacobi field $j(\mathbf{x}, t)$ obtained from the corresponding Lagrangian field $J(\mathbf{y}, t)$ by replacing $\mathbf{y}$ by $\mathbf{Y}(\mathbf{x}, t)$

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$$j(\mathbf{x}, t) = J(\mathbf{Y}(\mathbf{x}, t), t)$$

We transform the right-hand side of (1.2) using the relation

$$\delta(\mathbf{y} - \mathbf{Y}(\mathbf{x}, t)) = j(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{X}(\mathbf{y}, t)) \tag{1.3}$$

Substituting (1.3) into (1.2) and using the properties of the delta-function, we obtain

$$f_E(\mathbf{y}, j; \mathbf{x}, t) = jf_L(\mathbf{x}, j; \mathbf{y}, t) \tag{1.4}$$

where

$$f_L(\mathbf{x}, j; \mathbf{y}, t) = \langle\delta(\mathbf{x} - \mathbf{X}(\mathbf{y}, t))\delta(j - J(\mathbf{y}, t))\rangle \tag{1.5}$$

is the joint Lagrangian probability distribution of the coordinates $\mathbf{X}(\mathbf{x}, t)$ of a particle with Lagrangian coordinate $\mathbf{y}$ and Lagrangian Jacobi field $J(\mathbf{y}, t)$.

If the random field $\mathbf{v}(\mathbf{x}, t)$ of the medium is statistically homogeneous, then one more useful formula follows from (1.4), which relates the Lagrangian and Eulerian probability distributions of the Jacobian. We shall find it, observing that in a statistically homogeneous medium both probability distributions in (1.4) depend, apart from j, only on the difference of the spatial coordinates $\mathbf{x}$ and $\mathbf{y}$

$$f_E(\mathbf{y} - \mathbf{x}, j; t) = jf_L(\mathbf{x} - \mathbf{y}, j; t)$$

Integrating both sides of this equality with respect to all xs or ys, we find that the probability distribution of the Eulerian Jacobi field $f_E(j; t) = \langle\delta(j - j(\mathbf{y}, t))\rangle$ is equal to the probability distribution of the corresponding Lagrangian field $f_L(j; t) = \langle\delta(j - J(\mathbf{y}, t))\rangle$ with multiplier j, which takes into account the increase (for $j > 1$) of the share of expanding fluid particles in the statistical ensemble of Eulerian fields compared with the ensemble of Lagrangian fields.

Formula (1.4) relates the Lagrangian and Eulerian probability distributions of the Jacobian. In physical applications it is more natural to be equipped with a similar formula relating the probability distributions of the density of passive tracers. Its derivation is similar to that of (1.4). We take the Eulerian joint probability distribution of the Lagrangian coordinates and density

$$\varphi_E(\mathbf{y}, \rho; \mathbf{x}, t) = \langle\delta(\mathbf{y} - \mathbf{Y}(\mathbf{x}, t))\delta(\rho - \rho(\mathbf{x}, t))\rangle$$

Observing that (1.3) can be written in the equivalent form

$$\delta(\mathbf{y} - \mathbf{Y}(\mathbf{x}, t)) = \frac{\rho_0(\mathbf{y})}{R(\mathbf{y}, t)}\delta(\mathbf{x} - \mathbf{X}(\mathbf{y}, t))$$

where $\rho_0(\mathbf{y})$ is the initial density field and $R(\mathbf{y}, t)$ is the density field in the Lagrangian system of coordinates, we rewrite the previous equality in the form

$$\rho\varphi_E(\mathbf{y}, \rho; \mathbf{x}, t) = \rho_0(\mathbf{y})\varphi_L(\mathbf{x}, \rho; \mathbf{y}, t) \tag{1.6}$$

where

$$\varphi_L(\mathbf{x}, \rho; \mathbf{y}, t) = \langle\delta(\mathbf{x} - \mathbf{X}(\mathbf{y} - t))\delta(\rho - R(\mathbf{y}, t))\rangle \tag{1.7}$$

is the Lagrangian joint probability distribution of the fields $\mathbf{X}(\mathbf{y}, t)$ and $R(\mathbf{y}, t)$. It follows that (1.6) is the desired formula relating the Lagrangian and Eulerian density fields. In this formula, and everywhere below, the initial density $\rho_0(\mathbf{y})$ is assumed to be specified.

Integrating (1.6) with respect to all ys, we arrive at the formula

$$\rho\varphi_\rho(\rho; \mathbf{y}, t) = \int \rho_0(\mathbf{y})\varphi_L(\mathbf{x}, \rho; \mathbf{y}, t)d\mathbf{y} \tag{1.8}$$

which expresses the one-point probability distribution of the Eulerian density field $\varphi_\rho(\rho; \mathbf{x}, t) = \langle\delta(\rho - \rho(\mathbf{x}, t))\rangle$ in terms of the joint probability distribution of the Lagrangian fields $\mathbf{X}(\mathbf{y}, t)$ and $R(\mathbf{y}, t)$.

Integrating (1.8) with respect to ρ, we obtain the formula

$$\langle \rho(\mathbf{x},t) \rangle_E = \int \rho_0(\mathbf{y}) f_X(\mathbf{x};\mathbf{y},t) d\mathbf{y} \tag{1.9}$$

known as the fundamental theorem of the theory of turbulent diffusion [1]. It expresses the mean density of passive tracers in terms of the probability distribution of the coordinates of fixed particles

$$f_X(\mathbf{x};\, \mathbf{y},\, t) = \langle \delta(\mathbf{x} - \mathbf{X}(\mathbf{y},\, t)) \rangle \tag{1.10}$$

We will state one more, sometimes more convenient, modification of (1.8). The point is that the Lagrangian probability distribution φ_L on the right-hand side of (1.8) depends not only on the objective properties of the chaotically moving medium, but also on the "subjective" initial tracer density $\rho_0(\mathbf{y})$. To separate explicitly the contributions of the objective and subjective factors, we express φ_L in (1.7) by means of the "completely objective" Lagrangian distribution f_L given by (1.5) using the obvious equality

$$\delta\left(\rho - \frac{\rho_0(\mathbf{y})}{J(\mathbf{y},t)}\right) = \frac{\rho_0(\mathbf{y})}{\rho^2} \delta\left(J(\mathbf{y},t) - \frac{\rho_0(\mathbf{y})}{\rho}\right)$$

according to which

$$\varphi_L(\mathbf{x},\rho;\mathbf{y},t) = \frac{\rho_0(\mathbf{y})}{\rho^2} f_L\left(\mathbf{x},\frac{\rho_0(\mathbf{y})}{\rho};\mathbf{y},t\right)$$

From this and (1.8) we obtain

$$\varphi_\rho(\rho;\mathbf{x},t) = \frac{1}{\rho^3} \int \rho_0^2(\mathbf{y}) f_L\left(\mathbf{x},\frac{\rho_0(\mathbf{y})}{\rho};\mathbf{y},t\right) d\mathbf{y} \tag{1.11}$$

2. THE PROBABILITY PROPERTIES OF THE DENSITY IN AN INCOMPRESSIBLE MEDIUM

We begin the analysis of the probability properties of the density with the simple case of a chaotically moving incompressible fluid. Then the Lagrangian distribution on the right-hand side of the equality has the degenerate form

$$f_L(\mathbf{x}, j;\, \mathbf{y},\, t) = f_X(\mathbf{x};\, t|\mathbf{y})\delta(j - 1), \quad f_X(\mathbf{x};\, t|\mathbf{y}) = \langle \delta(\mathbf{x} - \mathbf{X}(\mathbf{y},\, t)) \rangle \tag{2.1}$$

where $f_X(\mathbf{x};\, t|\mathbf{y})$ is the probability distribution of the coordinates of a specified tracer particle.

Below we shall study the statistical properties of passive tracers in the diffusion approximation. As applied to the problem under consideration, this approximation assumes rapid decay of the correlations of the velocity field, so that the correlation tensor (1.1) can be replaced by the effective tensor $b_{ij}(\mathbf{s}, \tau) = d_{ij}(\mathbf{s})\delta(\tau)$ with a correlation time of zero. Then the distribution, determined by the second formula in (2.1), satisfies the standard diffusion equation [5–7]

$$\frac{\partial f_X}{\partial t} = D\Delta f_X, \quad f_X(\mathbf{x};t=0|\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \quad D = \frac{1}{N}\int_0^\infty b_{ii}(0,\tau)d\tau \tag{2.2}$$

Here D is the coefficient of turbulent diffusion and N is the dimension of the space. We observe that in the case of an incompressible medium the identity

$$f_X(\mathbf{x};\, t|\mathbf{y}) \equiv f_Y(\mathbf{y};\, t|\mathbf{x})$$

follows as a special case of the first formula in (2.1). Using this identity, we can rewrite the first formula in (2.1) in the form

$$f_L(\mathbf{x}, j;\, \mathbf{y},\, t) = f_Y(\mathbf{y};\, t|\mathbf{x})\delta(j - 1) \tag{2.3}$$

which proves to be more convenient in the analysis.

Substituting (2.3) into (1.11), the expression for the Eulerian probability distribution of the density in a turbulent incompressible medium can be transformed into

$$\varphi_\rho(\rho;\mathbf{x},t)=\frac{1}{\rho}\int\rho_0(\mathbf{y})f_Y(\mathbf{y},t|\mathbf{x})\delta(\rho-\rho_0(\mathbf{y}))d\mathbf{y} \tag{2.4}$$

We shall indicate some trivial consequences of this formula, which are, however, useful in what follows. For $t = 0$, when there is yet no mixing of the tracers due to the chaotic motion, $f_Y = \delta(\mathbf{y}-\mathbf{x})$, and the probability distribution (2.4) degenerates into a delta-shaped one: $\varphi_\rho(\rho;\mathbf{x},t) = \delta(\rho-\rho_0(\mathbf{x}))$. Such a degenerate distribution will be obtained at any instant of time if the initial tracer density is constant: $\rho_0(\mathbf{x}) = \rho_0 = \text{const}$. Then any motions of the incompressible fluid leave the tracer density invariant, and its probability distribution (2.4) is equal to $\varphi_\rho(\rho;\mathbf{x},t) = \delta(\rho-\rho_0)$. But if the initial tracer density is distributed non-uniformly in space, then in different samples particles with different initial (Lagrangian) coordinates $\mathbf{y}$, and so with different densities $\rho_0(\mathbf{y})$, will arrive at the observation point with Eulerian coordinates $\mathbf{x}$ because of the chaotic motion of the medium. As a result, the tracer density is random in the Euler representation and its probability distribution is no longer delta-shaped but "spreads out" along the ρ axis.

Using the "pricking out" property of the delta-function, we rewrite expression (2.4) for the probability distribution of the density field as a surface integral

$$\varphi_\rho(\rho;\mathbf{x},t)=\int_{S(\rho)}\frac{f_Y(\mathbf{y};t|\mathbf{x})}{|\nabla\rho_0(\mathbf{y})|}ds \tag{2.5}$$

where integration is over the level surface $S(\rho)$ of the initial density fields $\rho_0(\mathbf{y})$. The points $\mathbf{y}(\rho)$ of this surface satisfy the equality $\rho_0(\mathbf{y}(\rho)) = \rho$. With natural assumptions about the form of $\rho_0(\mathbf{y})$, (2.5) can be simplified even further. For example, suppose that the initial tracer distribution is spherical, i.e. $\rho_0 = \rho_0 = \rho_0(r)$, where r is the radial coordinate. Then the level surfaces will be spherical and (2.5) becomes

$$\varphi_\rho(\rho;\mathbf{x},t)=-r^2(\rho)\frac{dr}{d\rho}\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi f_Y(\mathbf{y};t|\mathbf{x}) \tag{2.6}$$

where θ and φ are angular coordinates on the sphere of radius $r(\rho)$, defined as the root of the equation $\rho_0(r) = \rho$. For simplicity we assume that $\rho_0(r)$ is a monotonically decreasing function, so that only one sphere corresponds to each value of ρ.

We substitute the solution of the diffusion equation (2.2) into (2.6)

$$f_Y(\mathbf{y};\mathbf{x},t)=\frac{1}{(4\pi Dt)^{3/2}}\exp\left[-\frac{\mathbf{y}-\mathbf{x}}{4Dt}\right] \tag{2.7}$$

and we orient the spherical system of coordinates in such a way that the direction $\theta = 0$ coincides with the direction towards the point of observation $\mathbf{x}$. Then

$$(\mathbf{x}-\mathbf{y})^2 = r^2(\rho) + R^2 - 2r(\rho)R\cos\theta$$

where $R = |\mathbf{x}|$ is the distance between the point of observation and the origin, and (2.6) after transformations and integration takes the form

$$\varphi_\rho(\rho;R,t)=-\frac{2r^2}{R\sqrt{4\pi Dt}}\frac{dr}{d\rho}\exp\left(-\frac{r^2+R^2}{4Dt}\right)\operatorname{sh}\frac{Rr}{2Dt} \tag{2.8}$$

It remains to substitute the explicit form of $r(\rho)$ for a specified profile of the initial density field. For example, we take the Gaussian profile

$$\rho_0(r) = \rho_m\exp(-r^2/l^2) \tag{2.9}$$

where ρ_m is the maximum value of the density and l is the effective radius of the patch of passive tracers. Then

$$r(\rho) = l\sqrt{\ln\left(\frac{\rho_m}{\rho}\right)}, \quad \frac{d\rho}{dr} = -\frac{2r\rho}{l^2}, \quad 2r\frac{dr}{d\rho} = -\frac{l^2}{\rho}$$

and expression (2.8) becomes

$$\varphi_\rho(\rho; R, t) = \frac{1}{\rho_m}\varphi(a, z, \gamma) \tag{2.10}$$

$$\varphi(a,z,\gamma) = \begin{cases} 0, & a \geqslant 1 \\ \dfrac{a^{1/\gamma-1}}{z\sqrt{\pi\gamma}}\exp\left(-\dfrac{z^2}{\gamma}\right)\operatorname{sh}\left(\dfrac{2z}{\gamma}\sqrt{\ln\dfrac{1}{a}}\right), & 0 < a < 1 \end{cases}$$

where a, z, γ are dimensionless variables

$$a = \frac{\rho}{\rho_m}, \quad z = \frac{R}{l} = \frac{|\mathbf{x}|}{l}, \quad \gamma = \frac{4Dt}{l^2} \tag{2.11}$$

Graphs of the probability distribution of the density field (2.10) for $z = 0.8$ and various values of γ are shown in Fig. 1.

Expression (2.10), which contains in compact form all the information on the one-point statistical characteristics of the tracer density field in an incompressible medium, is not very convenient if one is interested only in the moments of the density field and, in particular, its dispersion. We can obtain more convenient expressions for the moments of the Eulerian density field by multiplying the general expression (2.4) by ρ^n and integrating over all possible values of ρ. This gives

$$\langle\rho^n(\mathbf{x},t)\rangle_E = \int \rho_0^n(\mathbf{y})\, f_Y(\mathbf{y}; |\mathbf{x})\, d\mathbf{y} \tag{2.12}$$

In particular, for the Gaussian form of the initial tracer distribution (2.9) and probability distribution (2.7) of the coordinates we have

$$\langle\rho^n(\mathbf{x},t)\rangle_E = \frac{\rho_m^n}{(n\gamma+1)^{3/2}}\exp\left(-\frac{nz^2}{n\gamma+1}\right) \tag{2.13}$$

Here we use the dimensionless variables (2.11).

It follows from (2.13) that the dispersion of density fluctuations at the origin $\mathbf{x} = 0$ is equal to

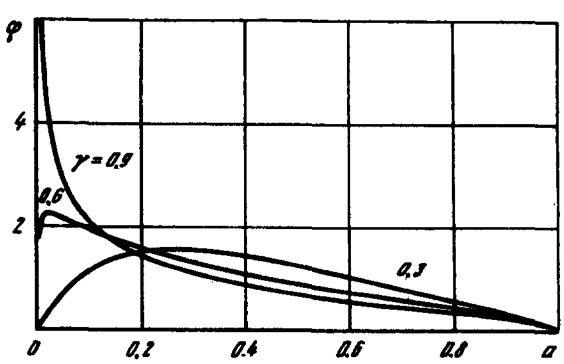

Fig. 1.

$$\sigma_\rho^2(x=0,t) = \langle \rho^2(0,t)\rangle_E - (\langle \rho(0,t)\rangle_E)^2 = \rho_m^2\left[\frac{1}{(1+2\gamma)^{3/2}} - \frac{1}{(1+\gamma)^3}\right] \tag{2.14}$$

3. THE PROBABILITY PROPERTIES OF THE DENSITY IN A COMPRESSIBLE MEDIUM

We will now analyse the probability properties of the Euler density field $\rho(\mathbf{x}, t)$ of "heavy" tracers in a compressible medium ignoring molecular diffusion. It is easiest to describe these properties in a Lagrangian system of coordinates, where the behaviour of the tracers is governed by a system of ordinary differential equations. In the case of "heavy" tracers the equations take the form

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}(\mathbf{X},t), \quad \mathbf{X}(\mathbf{y},t=0) = \mathbf{y}; \quad \frac{dJ}{dt} = u(\mathbf{X},t)\, J, \quad J(\mathbf{y},t=0) = 1 \tag{3.1}$$

where $u(\mathbf{X}, t) = \nabla\mathbf{v}(\mathbf{X}, t)$ is an auxiliary scalar field.

From the stochastic equations (3.1) in the diffusion approximation [5–7] one can change to a closed equation (similar to (2.2)) of the Kolmogorov type for the Lagrangian probability distribution (1.5)

$$\frac{\partial f_L}{\partial t} = D\Delta f_L + B\frac{\partial^2}{\partial j^2}(j^2 f_L) \tag{3.2}$$

$$f_L(\mathbf{x}, j; t=0|\,\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})\,\delta(j-1), \quad B = -\int_0^\infty \left(\frac{\partial^2}{\partial s_i^2} b_{ii}(\mathbf{s},\tau)\right)\Bigg|_{\mathbf{s}=0} d\tau$$

From the form of Eq. (3.2) and its initial conditions it follows that its solution can be represented in the form of two factors

$$f_L(\mathbf{x}, j; t|\,\mathbf{y}) = f_X(\mathbf{x}; t|\,\mathbf{y})\, f_J(j;t) \tag{3.3}$$

the first of which is the probability distribution of the coordinates of a specified particle of the tracer defined by the second formula of (2.1) and which satisfies the standard diffusion equation (2.2). The second factor, which describes the probability properties of the compressions and extensions of an infinitesimal volume of the tracers, satisfies the equation

$$\frac{\partial f_J}{\partial t} = B\frac{\partial^2}{\partial j^2}(j^2 f_J), \quad f_J(j; t=0) = \delta(j-1) \tag{3.4}$$

It can be integrated as the Kolmogorov equation for the transient probability distribution of an auxiliary Markov process $J(t)$ satisfying the stochastic equation

$$\frac{dJ}{dt} + BJ = u(t)\, J, \quad J(t=0) = 1 \tag{3.5}$$

where $u(t)$ is Gaussian white noise with correlation function $\langle u(t)\, u(t+\tau)\rangle = 2B\delta(\tau)$. It is natural to assume that the auxiliary process $J(t)$ and the compression-extension process $J(\mathbf{y}, t)$ under consideration are statistically equivalent. Therefore, we will study the behaviour of samples of the auxiliary process $J(t)$, assigning its properties to the actual process $J(\mathbf{y}, t)$.

The solution of the stochastic equation (3.5) has the form

$$J(t) = \exp[-Bt + \omega(t)] \tag{3.6}$$

where $\omega(t)$ is a Wiener process that such $\langle\omega(t)\rangle = 0$, $\langle\omega^2(t)\rangle = 2Bt$. From (3.6) as well as from (3.4) it follows that the probability distribution of $J(t)$ (and so that of $J(\mathbf{y}, t)$ also) is too normal

$$f_J(j;t) = \frac{1}{2j\sqrt{\pi Bt}}\exp\left[-\frac{\ln^2(je^{Bt})}{4Bt}\right] \quad (j>0) \tag{3.7}$$

with integral distribution function

$$F_J(j;t)=\int_0^j f_J(j;t)\,dj=\Phi\left[\frac{\ln(je^{Bt})}{2\sqrt{Bt}}\right],\quad \Phi(z)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{z}e^{-y^2}dy$$

Consider the time dependence of the statistical moments of $J(t)$. It follows from (3.4) that the nth moment $\langle J^n(t)\rangle_L$ satisfies the closed equation

$$\frac{d}{dt}\langle J^n\rangle_L=Bn(n-1)\langle J^n\rangle_L,\quad \langle J^n(t=0)\rangle_L=1 \tag{3.8}$$

the solution of which is

$$\langle J^n(t)\rangle_L=\exp[n(n-1)\,Bt] \tag{3.9}$$

We will now consider the general expression (1.11), substituting into its right-hand side the joint Lagrangian distribution (1.5) of the coordinates and the Jacobian in the compressible medium in place of (2.1). Using the statistical independence of the fluctuations of the Jacobian and the coordinates of a specified particle, i.e. the fact that (1.5) splits into the product of the probability distributions of **X** and J, we shall write the expression for the probability distribution of the Eulerian density field in a form close to (2.4)

$$\varphi_\rho^c(\rho;\mathbf{x},t)=\frac{1}{\rho}\int\rho_0(\mathbf{y})\,f_X(\mathbf{x};t|\,\mathbf{y})\left\langle\delta\left(\rho-\frac{\rho_0(\mathbf{y})}{J(t)}\right)\right\rangle d\mathbf{y} \tag{3.10}$$

The superscript c means that φ_ρ^c is the probability distribution of the density in the compressible medium and the angle brackets denote averaging over the ensemble of values of the auxiliary stochastic process $J(t)$ with probability distribution (3.7).

From (3.10) one can see that the probability distribution of the Eulerian density field in a turbulent compressible medium can be obtained by averaging over the statistics $J(t)$ of the probability distribution of the density in an incompressible medium

$$\varphi_\rho^c(\rho;\mathbf{x},t)=\langle J(t)\,\varphi_\rho(\rho;\mathbf{x},t|\,\rho_0\,/\,J(t))\rangle \tag{3.11}$$

Here $\varphi_\rho(\rho;\mathbf{x},t\,|\,\rho_0/J(t))$ is the probability distribution of the density in an incompressible medium subject to the condition that the initial tracer density is equal to $\rho_0(\mathbf{x})/J(t)$. In particular, for the initial density (2.9) we obtain

$$\varphi_\rho^c(\rho;\mathbf{x},t)=\frac{1}{\rho_m}\langle J(t)\,\varphi(aJ(t),z,\gamma)\rangle$$

where $\varphi(aJ(t), z, \gamma)$ is given by (2.10). Correspondingly, the moments of the probability distribution (3.10)

$$\langle\rho^n(\mathbf{x},t)\rangle_E=\langle J^{1-n}(t)\rangle\int\rho_0^n(\mathbf{y})\,f_X(\mathbf{x};t|\,\mathbf{y})\,d\mathbf{y} \tag{3.12}$$

differ from the same moments (2.12) of the density in an incompressible medium solely by a multiplier, which, by (3.9), is equal to

$$\langle J^{1-n}(t)\rangle=\exp[n(n-1)\,Bt] \tag{3.13}$$

It follows, for example, that the dispersion of the fluctuations of the Euler density field at the origin $\mathbf{x}=0$ and in the case of the initial density field (2.9) is equal to

$$\sigma_\rho^2(0,t)=\rho_m^2\left[\frac{e^{\beta\gamma}}{(1+2\gamma)^{3/2}}-\frac{1}{(1+\gamma)^3}\right],\quad \beta=\frac{Bl^2}{D} \tag{3.14}$$

Here we have introduced a new dimensionless parameter β, which describes the competition between the turbulent diffusion processes characteristic both for a compressible and an incompressible medium and the fluctuations of the Jacobian inherent only in the compressible medium.

Typical graphs of the dispersion (3.14) for various values of β depending on the dimensionless time γ are shown in Fig. 2. The dashed curve represents the graph of the dispersion (2.14) in the incompressible medium. Its form is determined by pure diffusion. Thus, the convergence of the dispersion to zero as $\gamma \to \infty$ is simply a consequence of the convergence to zero of all the moments of the density "spread out" (for $\gamma \gg 1$) over a large spatial domain (of radius $Dt \gg l^2$) and exceeding many times the initial "tracer patch" of size l. For $\beta \ll 1$ the pure diffusion at the origin dominates the fluctuations of the Jacobian, which are still small, and the graph of the dispersion of the density coincides with the dashed curve for a long time. As time passes, chaotic compressions and extensions begin to predominate and the dispersion begins to increase with t. This applies to an even greater extent to the behaviour of the dispersion for intermediate values ($\beta \approx 1$) and large values $\beta \gg 1$ of β.

Therefore, we shall first study the effects of chaotic compressions and extensions of the medium which predominate at an early or late stage. It is best to do this in the "ideal" case of a homogeneous initial density

$$\rho_0(\mathbf{x}) = \rho_0 = \text{const} \tag{3.15}$$

when "pure" diffusion has no effect on the density. Then the moments of the Eulerian density field (3.12) takes the form

$$\langle \rho^n(\mathbf{x},t)\rangle = \rho_0^n \langle J^{1-n}(t)\rangle = \rho_0^n \langle J^n(t)\rangle \tag{3.16}$$

(the last equality in the chain follows from (3.9)).

Equality (3.16) implies that not only the moments but also the probability distribution $J(t)$ are identical with the distribution of the normalized random density ρ/ρ_0. Therefore the equality

$$\rho(\mathbf{x}, t) = \rho(t) = \rho_0 J(t) \tag{3.17}$$

holds in the statistical sense. This is paradoxical in that it equates the density ρ and the Jacobian, which is the inverse of ρ in the arithmetic sense ($\rho \sim 1/J$). This paradox can be explained by the fact that the change from Lagrangian to Eulerian statistics of the same field (say, the density $\rho(\mathbf{x}, t)$ and $R(\mathbf{y}, t)$) is accompanied by a deformation of the probability measures: the compressed spatial domains have lower statistical weight in the Eulerian ensemble than in the Lagrangian one. As a result, the same fields have different statistics in different representations, while different fields have the same statistics. In particular, the Lagrangian density field J, the inverse Eulerian Jacobian $1/j$ and the statistically homogeneous normalized density field ρ/ρ_0, which is equal to the latter, all have the same probability properties.

On the basis of (3.17) we emphasize that even though by (3.17) and (3.9) the moments of the density increase exponentially with time

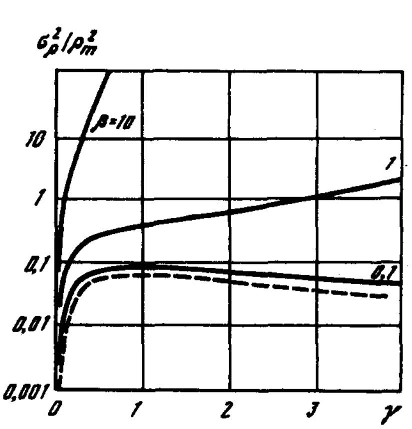

Fig. 2.

$$\langle \rho^n(\mathbf{x},t)\rangle_E = \rho_0^n \exp(n(n-1)\,Bt) \tag{3.18}$$

the probability that the random field $\rho(\mathbf{x}, t)$ exceeds the constant mean level $\langle\rho\rangle = \rho_0$ given by

$$P(J(\mathbf{y},t)>1) = \Phi\left(-\frac{\sqrt{Bt}}{2}\right) \approx \frac{2}{\sqrt{\pi Bt}}\exp\left(-\frac{Bt}{4}\right) \quad (t \to \infty)$$

tends exponentially to zero when $t \gg 1/B$. Extending the analogy between the time dependence of $\rho(\mathbf{x}, t)$ and $J(t)$, we can conclude that their realizations behave in the same way. Namely, practically all realizations $\rho(\mathbf{x}, t)$ lie under the majorizing curves $\rho_0 M(t)$: $M(t) = A\exp(-rBt)$ ($A > 1$ and $0 < r < 1$) and ultimately decay to zero [8–10]. But the exponential growth of the moments in the Eulerian plane (3.18) is due to the presence of pronounced peaks of the density $\rho \gg \rho_0$ in some realizations. Physically this means that the majority of stationary data transmitters will eventually turn out to be in domains in which the tracers are scarce ($\rho \ll \rho_0$) and only a few of them will lie inside a "macroparticle", i.e. a compact domain of high density and will register large values $\rho \gg \rho_0$ of the tracer density.

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